TWO THEOREMS ABOUT PROJECTIVE SETS[1]

BY

ALEXANDER S. KECHRIS AND YIANNIS N. MOSCHOVAKIS[†]

ABSTRACT

In this paper we prove two (rather unrelated) theorems about projective sets. The first one asserts that subsets of $\aleph_1$ which are Σ^1_2 in the codes are constructible; thus it extends the familiar theorem of Shoenfield that Σ^1_2 subsets of ω are constructible. The second is concerned with largest countable Σ^1_{2n} sets and establishes their existence under the hypothesis of Projective Determinacy and the assumption that there exist only countably many ordinal definable reals.

1. Preliminaries

Let $\omega = \{0, 1, 2, \cdots\}$ and $\mathscr{R} = {}^{\omega}\omega =$ the set of *reals*. We use $\alpha, \beta, \gamma, \cdots$ as variables over $\mathscr{R}$. The *product spaces* are $\mathscr{X} = X_1 \times \cdots \times X_k$, where $X_i = \omega$ or $X_i = \mathscr{R}$. If $P \subseteq \mathscr{X}$, P is called a *pointset* and we write interchangeably

$$x \in P \Leftrightarrow P(x).$$

The classes $\Sigma^1_n, \Pi^1_n, \boldsymbol{\Sigma}^1_n, \boldsymbol{\Pi}^1_n$, etc. of pointsets are defined as usual; see e.g. [7] or [5] where further information about games and determinacy can be found. We write *Determinacy* (Γ), where Γ is a class of pointsets, to indicate that every set in Γ is determined and we put

$$AD \Leftrightarrow \textit{Every pointset is determined}.$$

Also let *Uniformization* $(\Gamma) \Leftrightarrow$ For every relation $P \subseteq \mathscr{R} \times \mathscr{X}$ in Γ, there exists a relation P^* in Γ such that $P^* \subseteq P$ and

$$\exists \alpha P(\alpha, x) \Leftrightarrow \exists ! \, \alpha P^*(\alpha, x).$$

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We work entirely in Zermelo-Fraenkel set theory with *dependent choices* ($ZF + DC$) where

(DC) $$\forall u \in x \exists v (u,v) \in r \Rightarrow \exists f \forall n (f(n), f(n+1)) \in r.$$

All other hypotheses are stated explicitly. We let OD be the class of all ordinal definable sets.

Finally we repeat for convenience some definitions from [6].

A *norm* on a pointset P is a function $\phi: P \twoheadrightarrow \lambda$, from P *onto* an ordinal λ, the *ength* of ϕ. We call ϕ a Γ-*norm*, where Γ is a class of pointsets, if there are relations $\leqq_\Gamma$, $\leqq_{\breve{\Gamma}}$ in Γ and $\breve{\Gamma} = \{\mathscr{X} - P : P \in \Gamma\}$ respectively, such that

$$P(y) \Rightarrow \forall x (x \leqq_\Gamma y \Leftrightarrow x \leqq_{\breve{\Gamma}} y \Leftrightarrow [P(x) \ \& \ \phi(x) \leqq \phi(y)])$$

A *scale* on a pointset P is a sequence $\{\phi_n\}_{n \in \omega}$ of norms on P such that the following *limit condition* holds:

If $x_0, x_1, \cdots \in P$, if $\lim_{i \to \infty} x_i = x$ and if for each n and all large enough i, $\phi_n(x_i) = \lambda_n$, then $x \in P$ and for each n, $\phi_n(x) \leqq \lambda_n$.

We call $\{\phi_n\}_{n \in \omega}$ a Γ-*scale* if there are relations $S_\Gamma(n,x,y)$, $S_{\breve{\Gamma}}(n,x,y)$ in Γ and $\breve{\Gamma}$ respectively, such that for each n,

$$P(y) \Rightarrow \forall x (S_\Gamma(n,x,y) \Leftrightarrow S_{\breve{\Gamma}}(n,x,y) \Leftrightarrow [P(x) \ \& \ \phi_n(x) \leqq \phi_n(y)]).$$

We say that a class of pointsets Γ has the *scale property* and we write *Scale* (Γ) if every set in Γ has a Γ-scale. The basic results in [6] state that

$$\mathit{Determinacy}(\Delta^1_{2n}) \Rightarrow \mathit{Scale}(\Pi^1_{2n+1}),$$

$$\mathit{Uniformization}(\Pi^1_{2n+1}).$$

2. Subsets of $\aleph_1$ which are constructible

For any real α put

$$\leqq_\alpha \ = \{(m,n) : \alpha(\langle m,n \rangle) = 0\},$$

$$LOR = \{\alpha : \leqq_\alpha \text{ is an ordering}\}$$

and

$$WO = \{\alpha : \leqq_\alpha \text{ is a well ordering}\}.$$

If $\alpha \in WO$, let

$$|\alpha| = \text{length of } \leqq_\alpha.$$

Then the mapping $\alpha \to |\alpha|$, for $\alpha \in WO$, provides a natural coding system for ordinals less than $\aleph_1$ and we define for any $A \subseteq \aleph_1$ the code set of A by

$$\mathit{Code}(A) = \{\alpha : |\alpha| \in A\}.$$

The question arises, which subsets of $\aleph_1$ are constructible in terms of the complexity of their code sets. In complete analogy with the result of Shoenfield about subsets of ω, we establish

THEOREM 1. *If $A \subseteq \aleph_1$ and Code(A) is Σ^1_2, then A is constructible.*

Since *WO* is a Π^1_1 set, this clearly implies that if $A \subseteq \aleph_1$ and *Code* (A) is Π^1_2 then A is also constructible. Moreover if Solovay's $0^{\#}$ exists, then $0^{\#}$ is a Δ^1_3 subset of ω which is not in L (see [10]). It is easy to see that *Code* $(0^{\#})$ is Δ^1_3, so our result is essentially best possible.

The weaker result, when *Code*(A) is Π^1_1; was known to Solovay and is implicit in [9]. Solovay's proof uses forcing and cannot be used (apparently) to establish the full result. Our result can be used to give an easy forcing free proof of Solovay's theorem that

$$AD \Rightarrow (\forall A \subseteq \aleph_1)(\exists \alpha)[A \in L[\alpha]].$$

Our use of closed games to avoid forcing traces to [4].

Before we proceed to the main argument, we state and prove a folk-type result concerning the absoluteness of closed games. Let $\mathscr{S}$ be a set of even finite sequences from a set A. We define the game $G_{\mathscr{S}}$ as follows:

I	II	I plays a_0, II plays b_0, I then plays
a_0	b_0	a_1, II plays b_0, etc., where all a_i, b_i
a_1	b_1	are in A.
$\vdots$	$\vdots$	Then I wins iff for some n,
		$(a_0, b_0, \cdots, a_n, b_n) \in \mathscr{S}$.

Clearly the game is open in I.

LEMMA. *Let M be a transitive model of ZF + DC containing all the ordinals. Let $A, \mathscr{S} \in M$ and assume A is well orderable in M. Then, I has a winning strategy in $G_{\mathscr{S}} \Leftrightarrow M \models I$ has a winning strategy in $G_{\mathscr{S}}$, and similarly for II. Moreover the player who has a winning strategy has a winning strategy (for the game in the world) which lies in M.*

PROOF. For each $(a_0, b_0, \cdots, a_n, b_n)$ consider the subgame $G_{\mathscr{S}}(a_0, b_0, \cdots, a_n, b_n)$ defined by:

I	II	I plays $c_0, c_1, \cdots, II$ plays $d_0, d_1, \cdots$ and
c_0	d_0	I wins iff for some m
c_1	d_1	$(a_0, b_0, \cdots, a_n, b_n)^\frown(c_0, d_0, \cdots, c_m, d_m) \in \mathscr{S}$.
$\vdots$	$\vdots$	

Then define

$$\mathscr{S}^0 = \mathscr{S}$$

$$\mathscr{S}^\xi = \{(a_0, b_0, \cdots, a_n, b_n) : \exists a_{n+1} \in A \, \forall b_{n+1} \in A \\ \exists \eta < \xi((a_0, b_0, \cdots, a_{n+1}, b_{n+1}) \in \mathscr{S}^\eta)\}.$$

Then for each ξ, $(a_0, b_0, \cdots, a_n, b_n) \in \mathscr{S}^\xi \Rightarrow I$ has a winning strategy in $G_{\mathscr{S}}(a_0, b_0, \cdots, a_n, b_n)$. Using this we show:

II has a winning strategy in $G \Leftrightarrow \forall \xi[(\;) \notin \mathscr{S}^\xi]$.

Proof. If II has a winning strategy in $G_{\mathscr{S}} = G_{\mathscr{S}}((\;))$, then I has no winning strategy in $G_{\mathscr{S}}((\;))$; thus for all ξ, $(\;) \notin \mathscr{S}^\xi$. Conversely assume that for each ξ, $(\;) \notin \mathscr{S}^\xi$. We describe a winning strategy for II in $G_{\mathscr{S}}$ as follows: If I plays a_0, II plays the least b_0 (in a fixed well ordering of A) such that $\forall \xi (a_0, b_0) \notin \mathscr{S}^\xi$. Such a b_0 exists, because otherwise for all b, there exists a ξ such that $(a_0, b) \in \mathscr{S}^\xi$. Let $g(b) =$ least such ξ and find $\xi_0 >$ all $g(b)$, $b \in A$. Then $\forall b \, \exists \xi < \xi_0 (a_0, b) \in \mathscr{S}^\xi$; thus $(\;) \in \mathscr{S}^{\xi_0}$, a contradiction. Similarly if I plays a_1, II picks the least b_1 such that $\forall \xi (a_0, b_0, a_1, b_1) \notin \mathscr{S}^\xi$, etc.

Since the above equivalence was proved under the assumption "$ZF + DC + A$ is well orderable" and since $\xi \to \mathscr{S}^\xi$ is clearly an absolute map and $M \supseteq$ Ordinals, it is immediate that "II has a winning strategy" is absolute for M; thus the same is true for "I has a winning strategy." Moreover the argument above clearly provides a winning strategy for II which lies in M and wins in the world; hence it will be enough, in order to complete the proof, to show that when I has a winning strategy, we can find one (who wins in the world also) in M. Notice that

$$I \text{ has a winning strategy} \Leftrightarrow \exists \xi[(\;) \in \mathscr{S}^\xi]$$

and check that the following is a winning strategy for I which lies in M. Put $\xi_0 =$ least ξ such that $(\;) \in \mathscr{S}^\xi$. If $\xi_0 = 0$, I has already won. If $\xi_0 > 0$, let I play the least a_0 such that for every b, $\exists \xi < \xi_0 (a_0, b) \in \mathscr{S}^\xi$. If now II plays b_0, let $\xi_1 =$ least $\xi < \xi_0$ such that $(a_0, b_0) \in \mathscr{S}^\xi$. If $\xi_1 = 0$, I has already won, otherwise let I play the least a_1 such that for all b, $\exists\xi < \xi_1(a_0, b_0, a_1, b) \in \mathscr{S}^{\xi}$, etc. (Notice that $\xi_0 > \xi_1 > \cdots$, so this cannot go on.) ⊣

We are now ready to give the

PROOF OF THEOREM 1. Let $A \subseteq \aleph_1$ and $\mathrm{Code}(A) = P \in \Sigma^1_2$. Then

$$\xi \in A \Leftrightarrow (\exists\alpha)(\alpha \in P \,\&\, |\alpha| = \xi).$$

Let $\alpha \in P \Leftrightarrow \exists\beta\, Q(\alpha,\beta) \Leftrightarrow \exists\beta(f(\alpha,\beta) \in WO)$, where $Q \in \Pi^1_1$ and $f\colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ is recursive and for all α, β, $f(\alpha,\beta) \in LOR$. Then

$$\xi \in A \Leftrightarrow \exists\alpha\exists\beta(f(\alpha,\beta) \in WO \,\&\, |\alpha| = \xi).$$

Consider the following game G_ξ:

I			II		
ξ_0	a_0	b_0	η_0	θ_0	k_0
ξ_1	a_1	b_1	η_1	θ_1	k_1
⋮	⋮	⋮	⋮	⋮	⋮
	α	β			

I and II play, as in the diagram, natural numbers and ordinals $< \aleph_1$ and II wins iff *for every n*,

either for some $i \leqq n$, $\xi_i \geqq \xi$ *or* all the following are true:

a. The mapping $i \to \eta_i$ $(i \leqq n)$ is order preserving on the part of $\leqq_{f(\alpha,\beta)}$ already determined by $((a_0, \cdots, a_n), (b_0, \cdots, b_n))$ (notice that f is continuous).

b. The mapping $i \to \theta_i$ $(i \leqq n)$ is order preserving on the part of $\leqq_\alpha$ already determined by $(a_0, \cdots, a_n)$ and $\theta_i < \xi$, for each $i \leqq n$.

c. If $k_i \leqq n$, then $\theta_{k_i} = \xi_i$ and if $\langle k_i, k_i \rangle = j \leqq n$, then $a_j = 0$.

Notice now the following: *For $\xi < \aleph_1$, $\xi \in A \Leftrightarrow II$ has a winning strategy in G_ξ.*

PROOF. Assume $\xi \in A$; let α, β be such that $f(\alpha,\beta) \in WO$ and $|\alpha| = \xi$ and let $i \to \eta_i$ be an order preserving map on $\leqq_{f(\alpha,\beta)}$ into $\aleph_1$ and $i \to \theta_i$ a mapping from ω into ξ such that its restriction to Field $(\leqq_\alpha)$ is an order preserving bijection onto ξ, with inverse g. Consider the following strategy for II in G_ξ and verify easily that it is winning: If I plays ξ_0, II plays $\alpha(0), \beta(0), \eta_0, \theta_0, g(\xi_0)$ (unless $\xi_0 \geqq \xi$ in which case II plays anything). If I plays ξ_1, II gives $\alpha(1), \beta(1), \eta_1, \theta_1, g(\xi_1)$, etc.

Conversely assume II has a winning strategy. Let I play $\xi_0, \xi_1, \cdots$ enumerating without repetitions ξ, i.e., $\xi = \{\xi_0, \xi_1, \xi_2, \cdots\}$. Then II plays by his winning strategy and produces $\alpha, \beta, (\eta_0, \eta_1, \cdots,)$, $(\theta_0, \theta_1, \cdots)$, $(k_0, k_1, \cdots)$ such that $i \to \eta_i$ is order preserving on $\leqq_{f(\alpha,\beta)}$, thus $f(\alpha,\beta) \in WO$, $i \to \theta_i$ is order preserving on $\leqq_\alpha$ into ξ, thus $\alpha \in WO$, and finally $\xi_i \to k_i$ is an inverse to $i \to \theta_i$ on Field $(\leqq_\alpha)$, hence $|\alpha| = \xi$ and the proof is complete.

It is clear now that $G_\xi = G_{\mathscr{S}_\xi}$ where $\mathscr{S}_\xi$ is a set of finite sequences and moreover the map $\xi \to \mathscr{S}_\xi$ is absolute for L. Thus *for* $\xi < \aleph_1$,

$$\xi \in A \Leftrightarrow II \text{ has a winning strategy in } G_{\mathscr{S}_\xi}$$

$$\Leftrightarrow L \models II \text{ has a winning strategy in } G_{\mathscr{S}_\xi}.$$

So A is definable in L and therefore $A \in L$. Notice that the definition of A involves as the *only* parameter $\aleph_1$; thus $A = \tau^L(\aleph_1)$ for some term τ. ⊣

REMARK. 1. It is clear that the proof of Theorem 1 relativizes to any real α. Thus if we put

$$\tilde{L} = \{x : \exists\alpha(x \in L[\alpha])\}$$

we have $A \subseteq \aleph_1$ & $Code(A) \in \Sigma^1_2 \Rightarrow A \in \tilde{L}$. Kechris has proved a converse assuming $\forall\alpha(\alpha^\#$ exists) (see [10] for this notation), namely

$$\forall\alpha(\alpha^\# \text{ exists}) \Rightarrow \textit{Every } A \in L, A \subseteq \aleph_1 \textit{ has Code } (A) \in \Sigma^1_2.$$

Thus $\forall\alpha(\alpha^\#$ exists) implies

(*) $$A \subseteq \aleph_1 \Rightarrow (A \in \tilde{L} \Leftrightarrow Code\ (A) \in \Sigma^1_2).$$

A proof will appear in his thesis. Moreover (*) can be shown to be consistent with ZF using ideas of [3].

REMARK 2. Let $C \subseteq \mathscr{R}$ be a complete Π^1_1 set (i.e., for every $B \in \Pi^1_1$, there is a recursive $f: \mathscr{R} \to \mathscr{R}$ such that $\alpha \in B \Leftrightarrow f(\alpha) \in C$). Let ϕ be a Π^1_1-norm on C. Then ϕ has length $\aleph_1$ (see [5], p. 55). Thus it provides a coding system for ordinals $< \aleph_1$ and we can set for $A \subseteq \aleph_1$,

$$Code_\phi(A) = \{\alpha \in C : \phi(\alpha) \in A\}.$$

Then we can again establish

$$Code_\phi(A) \in \Sigma^1_2 \Rightarrow A \in L.$$

The reason is that if $Code_\phi(A) \in \Sigma^1_2$, then $Code(A) \in \Sigma^1_2$, since a simple computation shows that the relation

$$\alpha \in C \;\&\; \beta \in WO \;\&\; \phi(\alpha) = |\beta|$$

is a Σ^1_2 relation.

3. Largest countable Σ^1_{2n} sets

Solovay in [8] has proved the following theorem:

Assume $|\mathscr{R} \cap L| = \aleph_0$. *Then there exists a largest countable* Σ^1_2 *set of reals, namely* $\mathscr{R} \cap L$. Our next result extends this theorem to higher levels of the analytical hierachy.

THEOREM 2. *Assume* $|\mathscr{R} \cap OD| = \aleph_0$. *Then if Determinacy* (Δ^1_{2n}) *holds, there exists a largest countable* Σ^1_{2n+2} *set.*

Before we proceed to the proof let us establish some notation. Let κ be an ordinal. A *tree* T on $\omega \times \kappa$ is a set of finite sequences from $\omega \times \kappa$ such that if $u \in T$ and v is a subsequence of u then $v \in T$. For each such tree T we define the set of its branches by

$$[T] = \{(\alpha, f) \in {}^{\omega}\omega \times {}^{\omega}\kappa : \forall n((\alpha(0), f(0)), \cdots, (\alpha(n), f(n))) \in T\}$$

and we put

$$p[T] = \{\alpha : \exists f(\alpha, f) \in [T]\}.$$

Mansfield in [2] has proved the following theorem:

Assume T *is a tree on* $\omega \times \kappa$ *and* $A = p[T]$. *Then if* A *contains an element not in* $L[T]$, A *contains a perfect set.* (Mansfield used a forcing argument to prove his theorem; later Solovay gave a new forcing-free proof; for more details see [1].) Now let $A \subseteq \mathscr{R}$ and assume $\{\phi_n\}_{n \in \omega}$ is a scale on A. We define the tree T associated with this scale by

$$T = \{((\alpha(0), \phi_0(\alpha)), \cdots, (\alpha(n), \phi_n(\alpha))) : \alpha \in A\}.$$

Then $A = p[T]$.

PROOF OF THEOREM 2. By the main theorem in [6], *Determinacy*(Δ^1_{2n}) implies *Uniformization*(Π^1_{2n+1}). Thus for every countable Σ^1_{2n+2} set A, we can find a countable Π^1_{2n+1} set B, so that every real in A is recursive in some real in B. Thus it will be enough to find a countable Σ^1_{2n+2} set C which contains all countable Π^1_{2n+1} sets. Then $C^* = \{\alpha : \exists \beta(\beta \in C \,\&\, \alpha \text{ is recursive in } \beta)\}$ is the largest countable Σ^1_{2n+2} set.

Let $W \subseteq \omega \times \mathscr{R}$ be universal for Π^1_{2n+1} subsets of $\mathscr{R}$ and put

$$\alpha \in \mathscr{P} \Leftrightarrow (\alpha(0), \alpha') \in W,$$

where $\alpha' = (\alpha(1), \alpha(2), \cdots)$. Let also (by *Scale*$(\Pi^1_{2n+1})$) $\{\phi_n\}_{n \in \omega}$ be a Π^1_{2n+1}-scale on $\mathscr{P}$. Let T be the tree associated with this scale.

We now define C and show that it works:

$$\alpha \in C \Leftrightarrow (\exists m)(m^\frown\alpha \in \mathscr{P} \& |\{\beta: \phi_0(m^\frown\beta) \leqq \phi_0(m^\frown\alpha)\}| \leqq \aleph_0)$$

1. $C \in \Sigma^1_{2n+2}$

PROOF. Notice that

$$\alpha \in C \Leftrightarrow \exists m[m^\frown\alpha \in \mathscr{P} \& \exists\gamma\forall\delta[\phi_0(m^\frown\delta) \leqq \phi_0(m^\frown\alpha) \Rightarrow \exists k(\delta = (\gamma)_k)]]$$

2. *C contains every countable* Π^1_{2n+1} *set.*

PROOF. Let $B \in \Pi^1_{2n+1}$, $B \subseteq \mathscr{R}$, $|B| \leqq \aleph_0$. Find m such that $\beta \in B \Leftrightarrow (m, \beta) \in W \Leftrightarrow m^\frown\beta \in \mathscr{P}$. If $B \nsubseteq C$, let $\beta_0 \in B - C$. Put $\xi = \phi_0(m^\frown\beta_0)$. Then since $\beta_0 \notin C$, $|\{\beta: \phi_0(m^\frown\beta) \leqq \xi\}| > \aleph_0$; but $B \supseteq \{m^\frown\beta \in \mathscr{P}: \phi_0(m^\frown\beta) \leqq \xi\}$, a contradiction.

3. $C \subseteq L[T] \subseteq OD$; *thus* $|C| = \aleph_0$.

PROOF. It is enough to show that if for some m, ξ, $|\{\beta: \phi_0(m^\frown\beta) \leqq \xi\}| \leqq \aleph_0$, then $\{m^\frown\beta \in \mathscr{P}: \phi_0(m^\frown\beta) \leqq \xi\} \subseteq L[T]$. Put $T_{m,\xi} = \{((k_0, \xi_0), \cdots, (k_l, \xi_l)) \in T: k_0 = m \& \xi_0 \leqq \xi\}$. Clearly $T_{m,\xi} \in L[T]$ and the limit property of scales shows that $\alpha \in p[T_{m,\xi}] \Leftrightarrow \alpha \in \mathscr{P} \& \varphi_0(\alpha) \leqq \xi \& \alpha(0) = m$. Thus $\{m^\frown\beta \in \mathscr{P}: \varphi_0(m^\frown\beta) \leqq \xi\} = p[T_{m,\xi}]$ and so by Mansfield's Theorem

$$|\{\beta: \phi_0(m^\frown\beta) \leqq \xi\}| \leqq \aleph_0 \Rightarrow$$
$$\{m^\frown\beta \in \mathscr{P}: \phi_0(m^\frown\beta) \leqq \xi\} \subseteq L[T_{m,\xi}] \subseteq L[T]. \quad \dashv$$

We conclude with an open problem:

It is well known that every countable Σ^1_1 set contains only Δ^1_1 reals. Thus there is no largest countable Σ^1_1 set. Does either of these results generalize to Σ^1_{2n+1} $(n \geqq 1)$ under any reasonable hypotheses?

REMARK. After seeing a preliminary version of this paper, D. A. Martin observed that $|\mathscr{R} \cap OD| = \aleph_0$ can be replaced by Projective Determinacy in the statement of Theorem 2.

Added in proof: After the completion of this paper Kechris proved (assuming Projective Determinacy) the existence of largest countable Π^1_{2n+1} sets (for $n = 0$ this has been also proved independently by Sacks). He also proved (assuming *PD*) the non-existence of largest countable Σ^1_{2n+1} sets. Moreover Moschovakis has shown that countable Δ^1_{2n+1} sets contain only Δ^1_{2n+1} reals. If this is true for Σ^1_{2n+1} is still open.

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University of California, Los Angeles